Null Cones to Infinity, Curvature Flux, and Bondi Mass

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Mathematical General Relativity

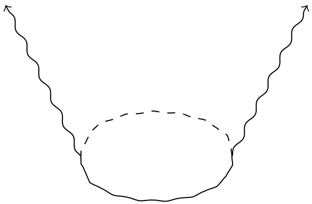
- In general relativity, spacetime is modeled as 4-dimensional *Lorentzian manifold* (M, g) satisfying the *Einstein equations*:

$$\operatorname{Ric}_g - \frac{1}{2} \operatorname{Scal}_g \cdot g = T.$$

  - Ric_g, Scal_g: Ricci and scalar curvature of (M, g).
  - T: *stress-energy tensor* for matter field.
- *Vacuum spacetimes*: no matter field ($T \equiv 0$)
  - *Einstein-vacuum equations*: $\operatorname{Ric}_g \equiv 0$.

Null Cones

- *Wave equation*, $\Box_g \phi = g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi \equiv 0$.
    - Can be thought of as linearized model for vacuum equations.
- *Null hypersurfaces*: induced metric is degenerate
    - Characteristics of the wave equation.
    - Generated by null geodesics.
- *Null cone*: null hypersurface $\mathcal{N}$ beginning from 2-sphere or point.
- *Curvature flux*: L^2-norm on $\mathcal{N}$ of certain components of R.
    - Important quantity in energy estimates.



Truncated null cone.

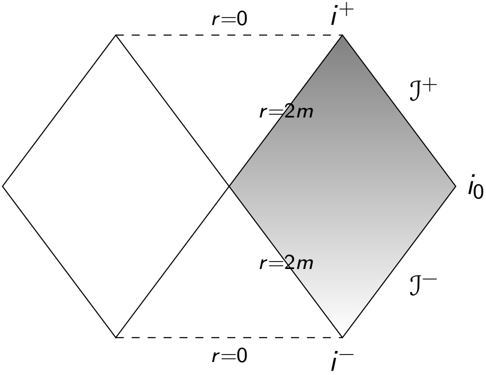
Schwarzschild Spacetimes

- *Schwarzschild spacetime*: spherically symmetric, black hole spacetimes
    - $m \geq 0$: "mass".
    - Satisfies Einstein-vacuum equations.
    - In the outer region $r > 2m$, metric can be expressed as

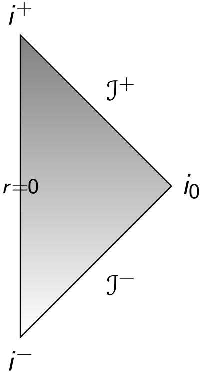
$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \mathring{\gamma}.$$

- $m = 0$: *Minkowski spacetime* $(-dt^2 + dr^2 + r^2 \mathring{\gamma})$.
- *Infinity*: represents faraway observer.
    - In these spacetimes, timelike/null/spacelike infinity can be explicitly constructed via *conformal compactification*.

Schwarzschild Spacetimes



Schwarzschild spacetime.



Minkowski spacetime.

Near-Minkowski Spacetimes

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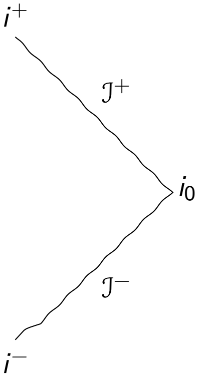
- Christodoulou-Klainerman: asymptotic stability of Minkowski spacetimes.
- Can recover similar structure at infinity as Minkowski spacetime.
- Stability of Schwarzschild, Kerr spacetimes: open problem.



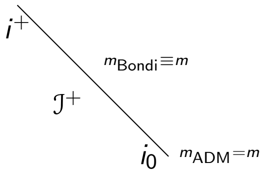
Near-Minkowski, at infinity.

Mass

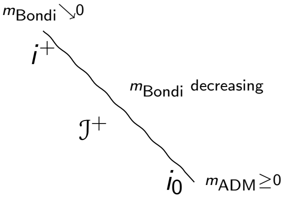
- In *asymptotically flat* spacetimes, with similar structures “at infinity”, there exist notions of total mass.
- *ADM mass*: applicable to spacelike hypersurfaces
  - Computed as limit at spacelike infinity.
  - Represents, e.g., total mass of initial data.
- *Bondi mass*: applicable to null cones
  - Computed as limit at a cut of null infinity.
  - Represents mass remaining in system after some has radiated away.

Mass

- Schwarzschild: static solution
  - $m_{\mathrm{ADM}}(\mathrm{init.}) = m$.
  - $m_{\mathrm{Bondi}} \equiv m$ on $\mathcal{I}^+$.

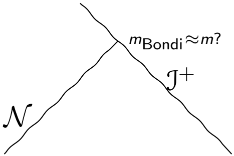


- Near-Minkowski: not static
  - *Positive mass thm.*: $m_{\mathrm{ADM}}(\mathrm{init.}) \geq 0$.
  - Mass loss: $0 \leq m_{\mathrm{Bondi}} \leq m_{\mathrm{ADM}}(\mathrm{init.})$.
  - $m_{\mathrm{Bondi}} \searrow 0$ in along $\mathcal{I}^+$.

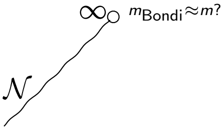


Main Goals

- Consider "near-Schwarzschild spacetime".
- "Eliminate all assumptions except at single infinite null cone."
  - (M, g): vacuum spacetime.
  - $\mathcal{N}$: future outgoing infinite null cone in (M, g).
  - $\mathcal{N}$ is "close to Schwarzschild null cone".



Intuitive picture



Assumed setting

Main Goals

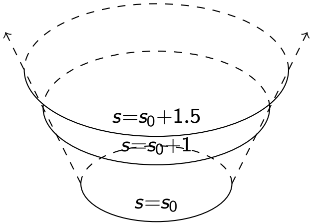
- Assume: $\mathcal{N}$ is "near-Schwarzschild".
  - "Weighted curvature flux" of $\mathcal{N}$ close to Schwarzschild.
  - "Initial data" of $\mathcal{N}$ close to Schwarzschild.
- Objective 1: control geometry of $\mathcal{N}$.
  - Quantitative bounds (for connection coefficients).
  - Asymptotic limits for coefficients at infinity.
- Objective 2: connection to physical quantities.
  - Control Bondi mass for $\mathcal{N}$.
  - Can also consider angular momentum, rate of mass loss.

Main Features

- No global assumptions on spacetime.
  - All assumptions on single null cone $\mathcal{N}$.
- Low-regularity quantitative assumptions.
  - At the level of curvature flux (L^2-norm of curvature on $\mathcal{N}$).
- Physical motivation.
  - What controls Bondi mass, etc.?
  - Requires finding “correct” foliation, i.e., approach to infinity.

Geodesic Foliations

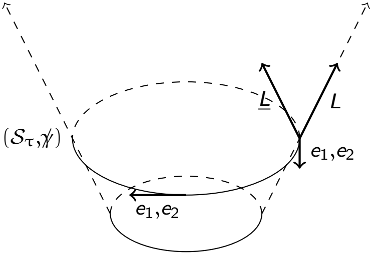
- *Geodesic foliation:* express $\mathcal{N}$ as one-parameter family of spheres.
  - Spheres determined by affine parameters of the null geodesics generating $\mathcal{N}$.
  - Algebraically simplest foliation.
- Write $\mathcal{N} \simeq [s_0, \infty) \times \mathbb{S}^2$.
  - s: affine parameter of null geodesics (starting from s_0).
  - s_0: radius of the initial sphere of $\mathcal{N}$.



Geodesic foliation.

Geodesic Foliations

- $\mathcal{S}_\tau$: level set $s = \tau$.
- $\gamma\!\!\!/$: induced metrics on the $\mathcal{S}_\tau$'s.
- Consider adapted *null frames*:
  - 2 spacelike directions e_1, e_2 tangent to $\mathcal{S}_\tau$.
  - 2 null directions normal to $\mathcal{S}_\tau$.
    - L tangent to $\mathcal{N}$ (and satisfies $Ls \equiv 1$)
    - $\underline{L}$ transverse to $\mathcal{N}$ (and satisfies $g(L, \underline{L}) \equiv -2$).



Null frame.

Formulation of Null Geometry

- Decompose spacetime curvature and connection quantities:
  - Spacetime curvature R:

$$\alpha_{ab} = R(L, e_a, L, e_b), \quad \beta_a = \frac{1}{2} R(L, \underline{L}, L, e_a), \quad \rho = \frac{1}{4} R(L, \underline{L}, L, \underline{L}),$$
$$\underline{\alpha}_{ab} = R(\underline{L}, e_a, \underline{L}, e_b), \quad \underline{\beta}_a = \frac{1}{2} R(\underline{L}, L, \underline{L}, e_a), \quad \sigma = \frac{1}{4} {}^\star R(L, \underline{L}, L, \underline{L}).$$

  - Connection coefficients:

$$\chi_{ab} = g(D_{e_a} L, e_b), \quad \underline{\chi}_{ab} = g(D_{e_a} \underline{L}, e_b), \quad \zeta_a = \frac{1}{2} g(D_{e_a} L, \underline{L}).$$

  - *Mass aspect function* (related to Hawking and Bondi mass):

$$\mu = -\gamma^{ab} \nabla_a \zeta_b - \rho + \frac{1}{2} \gamma^{ac} \gamma^{bd} \hat{\chi}_{ab} \underline{\hat{\chi}}_{cd}.$$

The Null Structure Equations

- The connection and curvature coefficients are related via a system of geometric PDE, called the *null structure equations*.
  - *Evolution equations*:

$$\nabla_L \chi \simeq -\chi \cdot \chi + \alpha, \quad \nabla_L \zeta \simeq \chi \cdot \zeta + \beta, \quad \nabla_L \underline{\chi} \simeq \rho + \nabla \zeta + \text{l.o.}, \quad \text{etc.}$$

  - *Elliptic equations*:

$$\mathcal{D} \hat{\chi} \simeq \beta + \text{l.o.}, \quad \mathcal{D} \zeta \simeq (\rho + \mu, \sigma) + \text{l.o.}, \quad \mathcal{K} \simeq -\rho + \chi \cdot \underline{\chi}, \quad \text{etc.}$$

  - *Null Bianchi equations*:

$$\nabla_L \beta \simeq \mathcal{D} \alpha + \chi \cdot \beta + \zeta \cdot \alpha, \quad \text{etc.}$$

- The vacuum equations are encoded within the structure equations.

Curvature Flux

- Define the *weighted curvature flux* for $\mathcal{N}$ to be

$$\mathcal{F}(\mathcal{N}) = \|s^2 \alpha\|_{L^2(\mathcal{N})} + \|s^2 \beta\|_{L^2(\mathcal{N})} + \|s\rho\|_{L^2(\mathcal{N})} + \|s\sigma\|_{L^2(\mathcal{N})} + \|\underline{\beta}\|_{L^2(\mathcal{N})}.$$

  - Generated as a local energy quantity from Bel-Robinson tensor.
  - *Bel-Robinson tensor*: "energy density" for spacetime curvature.
- Weights analogous to those found in C-K and K-N.
  - Note: s will be comparable to radii r of level spheres.

Hawking and Bondi Mass

- *Hawking mass* of $\mathcal{S}_\tau$:

$$\mathfrak{m}(\tau) = \frac{r}{2}\left[1 + \frac{1}{16\pi}\int_{\mathcal{S}_\tau} \operatorname{tr}\chi \operatorname{tr}\underline{\chi}\right] = \frac{r}{8\pi}\int_{\mathcal{S}_\tau} \mu.$$

  - r: area radius of $\mathcal{S}_\tau$.
- *If $r^{-2}\gamma\!\!\!/$ is asymptotically round*:
  - $\mathfrak{m}(\tau)$ converges to Bondi energy as $\tau \nearrow \infty$.

The Schwarzschild Case

- Standard outgoing shear-free null cones are:

$$\mathcal{N} = \{t - r^* = c,\ r \geq r_0\},$$

  - r^* is the "tortoise coordinate"

$$r^* = r + 2m \log\left(\frac{r}{2m} - 1\right).$$

- The affine parameter s on $\mathcal{N}$ is simply r.
- The null vector fields are

$$L = \left(1 - \frac{2m}{r}\right)^{-1} \partial_t + \partial_r, \qquad \underline{L} = \partial_t - \left(1 - \frac{2m}{r}\right) \partial_r.$$

The Schwarzschild Case

- Ricci coefficients:

$$\chi = r^{-1}\gamma\!\!\!/, \qquad \underline{\chi} = -r^{-1}\left(1 - \frac{2m}{r}\right)\gamma\!\!\!/, \qquad \zeta \equiv 0.$$

- Nonvanishing curvature coefficients:

$$\rho = -\frac{2m}{r^3}.$$

- Mass aspect function:

$$\mu = \frac{2m}{r^3}.$$

The Main Objectives, Revisited

1. "Get to infinity."
    - Assume: curvature flux of $\mathcal{N}$ close to Schwarzschild values.
    - Assume: connection coefficients near Schwarzschild values at $\mathcal{S}_{s_0}$.
    - Show: connection coefficients uniformly controlled on $\mathcal{N}$.
    - Show: limits of connection coefficients at infinity.
2. "Get the right infinity."
    - Infinity from Step 1 needs not correspond to Bondi mass.
    - Search instead for a "better" infinity.
    - Controlling "Bondi mass" and "angular momentum".

Theorem 1

Theorem (*Alexakis-S., 2012*: Control of Null Geometry)

Let $\mathcal{N}$ be as before, and assume the initial sphere $\mathcal{S}_{s_0}$ has radius $s_0 > 2m$. Assume the curvature flux bounds

$$s_0^{-\frac{3}{2}} \| s^2 \alpha \|_{L^2_s L^2_\omega} + s_0^{-\frac{3}{2}} \| s^2 \beta \|_{L^2_s L^2_\omega} + s_0^{-\frac{1}{2}} \| s (\rho + 2 m s^{-3}) \|_{L^2_s L^2_\omega} + s_0^{-\frac{1}{2}} \| s \sigma \|_{L^2_s L^2_\omega} + s_0^{\frac{1}{2}} \| \underline{\beta} \|_{L^2_s L^2_\omega} \leq C ,$$

and assume the following initial value bounds on $\mathcal{S}_{s_0}$:

$$s_0 \| \operatorname{tr} \chi - 2 s_0^{-1} \|_{L^\infty_\omega} + s_0^{\frac{1}{2}} \| \chi - s_0^{-1} \gamma \|_{H^{1/2}_\omega} + s_0^{\frac{1}{2}} \| \zeta \|_{H^{1/2}_\omega} \leq C ,$$

$$\| \underline{\chi} + s_0^{-1} \gamma (1 - 2 m s_0^{-1}) \|_{B^0_\omega} \leq C ,$$

$$s_0 \| \nabla (\operatorname{tr} \chi) \|_{B^0_\omega} + s_0 \| \mu - 2 m s_0^{-3} \|_{B^0_\omega} \leq C .$$

Theorem 1

Theorem (*Alexakis-S., 2012*: Control of Null Geometry)

If C is sufficiently small with respect to the "geometry of $\mathcal{S}$", then:

$$s_0^{-1} \| s^2 (\operatorname{tr} \chi - 2 s^{-1}) \|_{L^\infty_s L^\infty_\omega} \lesssim C,$$

$$s_0^{-\frac{1}{2}} \| s (\chi - s^{-1} \gamma) \|_{L^\infty_\omega L^2_s} + s_0^{-\frac{1}{2}} \| s \zeta \|_{L^\infty_\omega L^2_s} \lesssim C,$$

$$s_0^{-1} \| s^{\frac{3}{2}} (\chi - s^{-1} \gamma) \|_{L^4_\omega L^\infty_s} + s_0^{-1} \| s^{\frac{3}{2}} \zeta \|_{L^4_\omega L^\infty_s} \lesssim C,$$

$$s_0^{-\frac{3}{2}} \| \nabla_s [s^2 (\chi - s^{-1} \gamma)] \|_{L^2_s L^2_\omega} + s_0^{-\frac{3}{2}} \| \nabla_s (s^2 \zeta) \|_{L^2_s L^2_\omega} \lesssim C,$$

$$s_0^{-\frac{1}{2}} \| s \nabla \chi \|_{L^2_s L^2_\omega} + s_0^{-\frac{1}{2}} \| s \nabla \zeta \|_{L^2_s L^2_\omega} \lesssim C,$$

$$\| \underline{\chi} + s^{-1} (1 - 2 m s^{-1}) \gamma \|_{L^2_\omega L^\infty_s} \lesssim C,$$

$$s_0^{-1} \| s^2 \nabla (\operatorname{tr} \chi) \|_{L^2_\omega L^\infty_s} + s_0^{-1} \| s^2 (\mu - 2 m s^{-3}) \|_{L^2_\omega L^\infty_s} \lesssim C.$$

The Analysis

- Methods for controlling null geometry by curvature flux pioneered by Klainerman-Rodnianski (2005).
  - Finite geodesically foliated truncated null cones in vacuum.
  - Other variations (Q. Wang, Parlongue, S.)
- Application: breakdown criteria for Einstein equations, closely related to L^2-curvature conjecture.
- New generalizations and simplifications (S.).

The Analysis

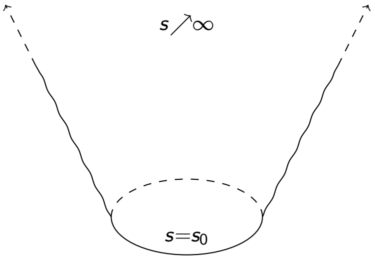
- Recall: assumed connection quantities are initially near-Schwarzschild.
- PDE problem: use curvature flux bounds and null structure equations to propagate connection estimates uniformly to all of $\mathcal{N}$.
    - Big bootstrap process.
    - Find and exploit structure of *null structure equations* on $\mathcal{N}$.
    - Evolution equations, elliptic equations, *null Bianchi equations*.
    - *New:* additional structures in Gauss-Codazzi equations.
- Propagation leads to limits of connection quantities at infinity.

Major Difficulties

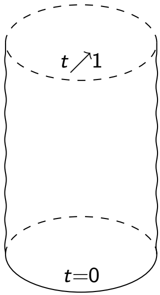
- Why is this hard?
- Low regularity (assuming only bounds for curvature on $\mathcal{N}$)
  - Canonical coordinate vector fields lack sufficient regularity.
  - Need Besov-type norms and estimates to close.
- Remedies:
  - Geometric tensorial Littlewood-Paley theory (heat flow, spectral).
  - Bilinear product and elliptic estimates (in Besov norms).
  - *New:* regular t-parallel frames (simplifies product estimates).
  - *New:* partial conformal smoothing (simplifies elliptic estimates).

Conversion to Small-Data Problem

- In PDEs: common to convert stability problem to small-data problem.
  - Consider as variables (weighted) deviations of curvature and connection coefficients from Schwarzschild values.
  - Convert outgoing infinite near-cone to finite near-cylinder.



Physical setting.



Renormalized setting.

The Renormalized System

- Renormalization of system has two main steps:
  1. Rescaling of metric: $\gamma = s^{-2}\not\gamma$ (expanding cone $\Rightarrow$ cylinder):
  2. Change of evolutionary variable

$$s \in [s_0, \infty) \Rightarrow t = 1 - \frac{s_0}{s} \in [0, 1).$$

- This transforms the natural covariant system (now w.r.t. γ and t):
  - Q. What are "natural" derivative operators to consider?
    - Spherical covariant derivatives to not change.
    - Elliptic operators are rescaled.
    - t-covariant derivatives ∇_t (general construction).

The Renormalized System

- Renormalized Ricci coefficients:

$$H = s_0^{-1}(\chi - s^{-1}\not\gamma), \quad Z = s_0^{-1} s\zeta, \quad \underline{H} = s^{-1}\underline{\chi} + s^{-2}(1 - 2ms^{-1})\not\gamma.$$

- Renormalized curvature coefficients:

$$A = s_0^{-2} s^2 \alpha, \quad B = s_0^{-2} s^3 \beta, \quad R = s_0^{-1}[s^3(\rho + i\sigma) + 2m], \quad \underline{B} = s\underline{\beta}.$$

- Renormalized mass aspect function:

$$M = s_0^{-1}(s^3\mu - 2m).$$

Why Renormalize?

1. The geometries of the spheres, w.r.t. γ, are nearly uniform.
   - Estimates on spheres have *common constants*.
   - Highlights the relevant quantities for controlling geometry.
2. The *weighted* inequalities in the main theorem become *unweighted* inequalities in the renormalized quantities.
   - Renormalized quantities expected to be uniformly $O(\epsilon)$.
3. Can reformulate null structure equations in renormalized system.
   - All analysis done on renormalized system.
4. Limits at infinity are w.r.t. renormalized quantities and γ.

Renormalized Main Theorem I

Theorem (Renormalized Formulation)

Let $\mathcal{N}$ be as before. Assume the curvature flux bounds

$$\|A\|_{L^2_t L^2_\omega} + \|B\|_{L^2_t L^2_\omega} + \|R\|_{L^2_t L^2_\omega} + \|\underline{B}\|_{L^2_t L^2_\omega} \leq C,$$

and assume the following initial value bounds on $\mathcal{S}_{s_0}$:

$$\|\operatorname{tr} H\|_{L^\infty_\omega} + \|(H, Z)\|_{H^{1/2}_\omega} + \|(\underline{H}, \nabla(\operatorname{tr} H), M)\|_{B^0_\omega} \leq C.$$

If C is sufficiently small with respect to the "geometry of $\mathcal{S}$", then:

$$\|\operatorname{tr} H\|_{L^\infty_t L^\infty_\omega} + \|(H, Z)\|_{N^1_{t,\omega} \cap L^\infty_x L^2_t \cap L^\infty_t H^{1/2}_\omega} \lesssim C,$$
$$\|(\underline{H}, \nabla(\operatorname{tr} H), M)\|_{L^\infty_t B^0_\omega \cap L^2_x L^\infty_t} \lesssim C.$$

Moreover, the geometries of the level spheres of $\mathcal{N}$ "remain regular".

Limits at Infinity

- Can refine renormalized theorem to produce limits at infinity.
  - $\nabla_t F$ is integrable on $\mathcal{N} \Rightarrow F$ is controlled uniformly on every level sphere and has a limit at infinity.
- Limiting geometry:
  - γ converges to a metric as $t \nearrow 1$ (i.e., $s \nearrow \infty$).
  - Weaker (L^2-type) convergence for Christoffel symbols, connection.
- Limiting quantities: $(H, Z, \underline{H}, M)$
  - Regularity is propagated from $\mathcal{S}_{s_0}$ to infinity.

Extension to Infinity

Corollary (*Alexakis-S., 2012*: Limits at Infinity)

Assume the same as before.

- γ *has a limit as* $s \nearrow \infty$ *(in* C^0 *and* H^1*).*
- H, Z, $\underline{H}$, M *have limits (with respect to* γ*) as* $s \nearrow \infty$*. These limits can be controlled (in same norms as initial condition) by* C*.*
- *Furthermore, with respect to* $\not\gamma$ *and* s*, the Hawking masses*

$$\mathfrak{m}(s) = \frac{r(s)}{2} \left[1 + \frac{1}{16\pi} \int_{\mathcal{S}_s} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} \right]$$

of the level spheres have a limit $\mathfrak{m}(\infty)$ *as* $s \nearrow \infty$*. Moreover,*

$$|\mathfrak{m}(\infty) - m| \lesssim C.$$

Bilinear Product Estimates

- To prove tensorial product estimates, one generally:
  - Decomposes into local scalars via coordinate fields.
  - Applies Euclidean product estimates.
  - Reconstructs tensorial estimates.
- Problem: transported coordinate fields barely lack enough regularity.
- Solution: t-parallel frames are in fact more regular.
- Observation: This is due to structure in Codazzi equations:

$$\operatorname{curl} \chi \simeq \beta + \text{l.o.t.}, \qquad \operatorname{curl} H \simeq B + \text{l.o.t.}.$$

  - Propagation of transported coordinate frames depend on $\nabla \chi \Rightarrow$ "loss of half a derivative".
  - Propagation of t-parallel frames depend only on $\operatorname{curl} \chi \Rightarrow$ "no loss".

(Besov) Elliptic Estimates

- Want elliptic estimates of the form:

$$\|\nabla \mathcal{D}^{-1} \xi\|_{B^0} \lesssim \|\xi\|_{B^0}.$$

  - $\mathcal{D}$: elliptic Hodge operator.
  - B^0: (geometric) zero-order Besov norm.
- Problem: Gauss curvatures of spheres are $H^{-\frac{1}{2}}$.
  - Proofs are technical, and result in additional error terms.
- Solution: partial conformal smoothing method
- Observation: decomposition of Gauss curvature as $L^2 + \operatorname{div} H^{\frac{1}{2}}$.
  - By conformal transform, can remove divergence from curvature.
  - Working with L^2 Gauss curvature, proofs are greatly simplified.
  - Removes previous error terms.

Avoidance of Infinite Decompositions

- Problem: previous proofs required elaborate infinite decompositions.
- Solution: bootstrap using "sum" norms.
- Observation: exact decomposition does not matter, only need *some* decomposition with required estimates.
    - Standard "sum" norms capture exactly this situation.

Obtaining the Bondi Energy

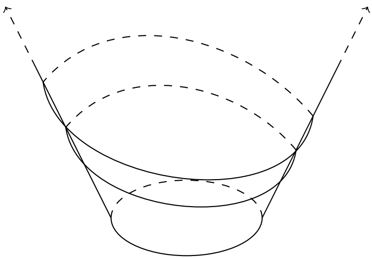
- If the limiting sphere of $\mathcal{N}$ (w.r.t. γ) is round/Euclidean:
  - The Hawking masses of the spheres converge to the Bondi energy.
- Problem: in our case, limiting sphere needs not be Euclidean.
- Gauss curvatures of spheres $(\mathcal{S}_s, \gamma)$ given by

$$\mathcal{K} = 1 - \frac{1}{2} \operatorname{tr} \underline{H} + O(s^{-1}).$$

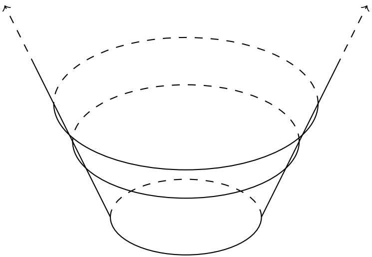
  - The Gauss curvatures have (very weak) limit at infinity.
  - This limit needs not be 1.

Finding the Correct Infinity

- Goal: find family of *asymptotically round* spheres going to infinity.
- Mechanism for obtaining spheres: *change of geodesic foliations*.
  - Can rescale parameter of each null generator by constant.
  - Change of foliation given by *distortion function* $v : \mathbb{S}^2 \to \mathbb{R}$.
  - e^v maps each null generator to scaling factor.



Decent foliation.



Great foliation.

Changes of Geodesic Foliations

- Transformation defined by relations:
  - Rescale tangent null vector field: $L' = e^{v} L$.
  - Change of affine parameter: $(s' - s_0) = e^{-v}(s - s_0)$.
- Other quantities also change by explicit formulas:
  - Null frame elements: $e_1, e_2, \underline{L}$
  - Connection coefficients: χ, $\underline{\chi}$, ζ
  - Curvature coefficients: α, β, ρ, σ, $\underline{\beta}$
  - Similarly for the renormalized quantities:

$$\gamma, t, H, Z, \underline{H}, A, B, R, \underline{R}.$$

The Main Idea

- From the Gauss equation:

$$\mathcal{K}' = 1 - \frac{1}{2} \operatorname{tr}' \underline{H}' + O(s'^{-1}).$$

- Goal: find change of foliation v so that $\operatorname{tr}' \underline{H}'$ vanishes.
- From change of foliation formulas (long computation),

$$\operatorname{tr}' \underline{H}' = \operatorname{tr} \underline{H} + 2\Delta_\gamma v + 2(e^{2v} - 1) + O(s^{-1}).$$

- Problem becomes an elliptic equation *at infinity*:

$$\Delta_{\gamma_\infty} v + e^{2v} = 1 - \frac{1}{2} \operatorname{tr}_\infty \underline{H}_\infty = \mathcal{K}_\infty.$$

  - Closely related to the uniformization theorem.

Main Difficulties

1. Want *smooth* family of spheres.
   - Obtain family v_y of refoliations, with $v_y \to v$.
   - Solve approximate PDE for v_y on $\mathcal{S}_y$.
   - Choose level sphere $s'_y = y$ from each v_y-foliation.
2. $\mathcal{K}$ converges too weakly $(H^{-\frac{1}{2}})$ to infinity.
   - Our desired v is not regular enough for estimates.
   - Solution: partial conformal smoothing of γ.
   - Smoothes curvature from $H^{-\frac{1}{2}}$ to L^2.
3. Need convergence of Hawking masses as $y \nearrow \infty$.
   - Need uniform smallness for v_y's (in appropriate norms).
   - Need convergence $\lim_{y \nearrow \infty} v_y = v$ (in appropriate norms).

Construction of the v_y's

1. Partial conformal smoothing $\gamma \mapsto e^{2u}\gamma$.
   - Removes worst term from $\mathcal{K} \Rightarrow$ curvature now in L^2.
   - Conformal factor $u \to 0$ as $s \nearrow \infty$.
   - u is *not* included in our construction of v.
2. More partial conformal smoothing.
   - Technique applied by L. Bieri.
   - Smoothes Gauss curvature from L^2 to L^∞.
3. Uniformization problem.
   - Final part of v obtained by solving uniformization problem on smoothed spheres with L^∞-curvature.
   - Technique from Christodoulou-Klainerman.

Main Theorem II

Theorem (*Alexakis-S., 2012: Control of Bondi mass*)

Assume the same as before. Then,

$$|m_{Bondi}(\mathcal{N}) - m| \lesssim C.$$

Similar estimates hold for angular momentum and rate of mass loss.

The End

Thank you for your attention!