Breakdown Criteria for Nonvacuum Einstein Equations

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The Breakdown Problem

- General question: *Under what conditions can an existing local solution of an evolution equation on a finite interval* $[0, T)$ *be further extended past* T*?*
- Why is this useful?
  1. Characterize breakdown of solutions.
  2. Global existence problem.

Nonlinear Wave Equations

- Equations of the form

$$\Box\phi = (\partial\phi)^2, \qquad \phi|_{t=0} = \phi_0, \qquad \partial_t\phi|_{t=0} = \phi_1.$$

  - Local existence for H^s-spaces.
  - If local solution on $[0, T)$ satisfies

    $$\|\partial\phi\|_{L^\infty} < \infty, \tag{1}$$

    then solution can be extended past T.
  - Time of existence controlled by H^s-norms, which can be uniformly controlled on $[0, T)$ using (1).

Incompressible 3-d Euler equations

- $u : \mathbb{R}^{1+3} \to \mathbb{R}^3$, $p : \mathbb{R}^{1+3} \to \mathbb{R}$.

$$\partial_t u + u \cdot \nabla u + \nabla p = 0,$$
$$\nabla \cdot u = 0.$$

  Vorticity: $\omega = \nabla \times u$.
- Beale, Kato, Majda (1984): If a local solution has ω bounded in $L^1_t L^\infty_x$, then it can be extended.
  - Need not bound all of ∇u.

Yang-Mills Equations

- Eardley, Moncrief (1982): global existence in $\mathbb{R}^{1+3}$.
  - Continuation criterion: $\|F\|_{L^\infty} < \infty$
  - F - Yang-Mills "curvature".
  - $\|F\|_{L^\infty}$ controlled using wave equations and fundamental solutions.
- Chruściel, Shatah (1997): generalized to globally hyperbolic $(1+3)$-dim. Lorentz manifolds.

Results for Vacuum Equations

- Einstein vacuum: $(1+3)$-dim. spacetimes (M, g),

$$\mathrm{Ric}_g = 0.$$

- Anderson (2001): $\|R_g\|_{L^\infty} < \infty \Rightarrow$ solution can be extended.
  - Geometric, requires two derivatives of g.
- Other continuation criteria:

$$\|\partial g\|_{L^\infty} < \infty, \text{ or } \|\partial g\|_{L^1_t L^\infty_x} < \infty.$$

  - Not geometric, depends on choice of coordinates.

Improved Results

- Klainerman, Rodnianski (2008): improved breakdown criterion for vacuum:

$$\|k\|_{L^\infty} + \|\nabla (\log n)\|_{L^\infty} < \infty$$

  - CMC foliation, compact time slices.
  - k, n - second fundamental form, lapse of time slices.
  - Geometric, do not need full coordinate system.
  - k and $\nabla(\log n)$ at level of ∂g, but do not cover all components of ∂g.
- D. Parlongue (2008): vacuum, maximal foliation, asymptotically flat time slices, replaced L^∞ by $L^2_t L^\infty_x$.
- Q. Wang (2010): vacuum, CMC, compact time slices, replaced L^∞ by $L^1_t L^\infty_x$.

General Einstein Equations

- Spacetime (M, g, Φ), Φ - matter fields.
- Einstein equations:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = Q_{\alpha\beta}.$$

Q - energy-momentum tensor.

- Einstein-scalar ($\Phi = \phi$ - scalar):

$$\Box_g \phi = 0, \qquad Q_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} \partial^\mu \phi \partial_\mu \phi.$$

- Einstein-Maxwell ($\Phi = F$ - 2-form):

$$D^\alpha F_{\alpha\beta} = 0, \qquad D_{[\alpha} F_{\beta\gamma]} = 0,$$

$$Q_{\alpha\beta} = F_{\alpha\mu} F_\beta{}^\mu - \frac{1}{4} g_{\alpha\beta} F^{\mu\nu} F_{\mu\nu}.$$

The Main Questions

- Does there exist a “breakdown criterion” similar to K-R for Einstein-scalar and Einstein-Maxwell spacetimes.
- Other nonvacuum settings?

The Basic Setting

- Same setting as K-R, but with E-S or E-M spacetime (M, g, Φ) rather than E-V.
  - Time foliation:

$$M = \bigcup_{t_0 < \tau < t_1} \Sigma_\tau, \qquad t_0 < t_1 < 0.$$

  - Σ_τ's are compact.
  - CMC foliation: $\operatorname{tr} k = \tau < 0$ on Σ_τ.

The Main Theorem

Theorem

Assume an Einstein-scalar or Einstein-Maxwell spacetime (M, g, Φ) in the setting of the previous slide. If

$$\sup_{t_0 \leqslant \tau < t_1} \left(\| k(\tau) \|_{L^\infty} + \| \nabla (\log n)(\tau) \|_{L^\infty} \right) < \infty, \qquad (2)$$

and the following bounds hold for the matter field,

$$\text{(E-S)} \qquad \sup_{t_0 \leqslant \tau < t_1} \| D\phi(\tau) \|_{L^\infty} < \infty, \qquad (3)$$

$$\text{(E-M)} \qquad \sup_{t_0 \leqslant \tau < t_1} \| F(\tau) \|_{L^\infty} < \infty, \qquad (4)$$

then (M, g, Φ) can be extended as a CMC foliation beyond time t_1.

Additional Remarks

- Strategy of proof analogous to K-R.
- We focus on E-M setting, since E-S is easier.
- The theorem extends to Einstein-Klein-Gordon and Einstein-Yang-Mills spacetimes (nontrivial).
- Result can likely be extended to $L_t^2 L_x^\infty$ and $L_t^1 L_x^\infty$ breakdown criteria.

Main Issues

- Presence of nontrivial Ricci curvature.
- Coupling between curvature and matter fields.
- E-M: New types of nonlinearities in wave equations for DF and curvature R.
  - Cannot be treated using methods of K-R.

The Cauchy Problem

- Given $(\Sigma_0, \gamma_0, k_0, \Phi_0)$, where
  - Σ_0 - Riemannian 3-manifold.
  - γ_0 - metric on Σ.
  - k_0 - symmetric 2-tensor ("second fundamental form").
  - Φ_0 - initial values for matter fields.
- Assume initial data satisfies *constraint equations*.
- Solve for spacetime (M, g, Φ), where $M \cong I \times \Sigma_0$:
  - (Σ_0, γ_0) imbedded as "initial" time slice of M, with second fundamental form k_0.
  - Φ_0 corresponds to value of Φ on Σ_0.

Local Well-Posedness

- Local existence: time of existence depends on

$$\mathfrak{E}_0 \sim \|k_0\|_{H^3} + \|\mathcal{R}_0\|_{H^2} + \|\Phi_0\|_{H^3} \,,$$

  and geometric properties of Σ_0.
  - $\mathcal{R}_0$ - Ricci curvature of Σ_0.
  - E-M: $\Phi_0 = (E_0, H_0)$ - electromagnetic decomposition
- Main goal: uniformly control analogous quantities $\mathfrak{E}(\tau)$ for each Σ_τ for all $t_0 < \tau < t_1$.
  - Apply local existence theorem to each Σ_τ.
- Elliptic estimates: suffices to uniformly bound *spacetime* quantities

$$\mathfrak{E}(\tau) \sim \|R(\tau)\|_{H^2} + \|F(\tau)\|_{H^3} \,.$$

Important Preliminaries

- Breakdown criterion $\Rightarrow$ the deformation tensor

$$^T\pi = \mathcal{L}_T g$$

is uniformly bounded (i.e. T "almost Killing").

  - T - future unit normal to Σ_τ's.
- Construct "energy-momentum tensors" similar to Q for scalar and Maxwell fields.
  - Generalized (tensorial) wave equations.
  - Generalized Maxwell-type equations.
- The above two ideas imply energy inequalities.

A Priori Energy Estimates

- Define
$$\mathcal{E}_0(\tau) = \|R(\tau)\|_{L^2} + \|DF(\tau)\|_{L^2}.$$
- Using generalized EMT's from R and F, we obtain
$$\mathcal{E}_0(\tau) \lesssim \mathcal{E}_0(t_0).$$
- Due to coupling, R and DF must be handled concurrently.

Higher Order Energy Quantities

- Define higher order energy quantities

$$\mathcal{E}_1(\tau) = \|DR(\tau)\|_{L^2} + \left\|D^2F(\tau)\right\|_{L^2},$$
$$\mathcal{E}_2(\tau) = \left\|D^2R(\tau)\right\|_{L^2} + \left\|D^3F(\tau)\right\|_{L^2}.$$

- R, DR, DF, D^2F satisfy covariant wave equations.
- Goal: show uniformly in τ,

$$\mathcal{E}_1(\tau) + \mathcal{E}_2(\tau) \leqslant C. \tag{5}$$

- Main difficulty: must also bound

$$\|R(\tau)\|_{L^\infty} + \|DF(\tau)\|_{L^\infty}.$$

Null Cones

- For $p \in M$, we can define past null cone $N^-(p)$ about p.
  - Near p, $N^-(p)$ is smooth and parametrized by $s \in (0, \infty)$ and $\omega \in \mathbb{S}^2$; call this portion $\mathcal{N}^-(p)$.
  - L - geodesic null tangent vector field.
  - Null frames L, $\underline{L}$, e_1, e_2 - locally defined w.r.t. spherical foliation of $\mathcal{N}^-(p)$.

A Priori Local Estimates

- A priori L^2 flux bounds for R and DF on $\mathcal{N}^-(p)$, again using EMT's.
- Flux does not control all components of R and DF.
  - R: excludes $R_{\underline{L}e_a\underline{L}e_b}$.
  - DF excludes $D_{\underline{L}} F_{\underline{L}e_a}$.
- Also need higher-order flux estimates for DR and $D^2 F$ on $\mathcal{N}^-(p)$.
  - Cannot control all components of DR and $D^2 F$.
  - Also need uniform bounds for R and DF.

Revisiting the Uniform Bound

- Recall: need uniform bounds for

$$\|R(\tau)\|_{L^\infty} + \|DF(\tau)\|_{L^\infty} .$$

- Main idea: R, DF satisfy system of wave equations:

$$\Box_g R \cong F \cdot D^2 F + (R + DF)^2 + l.o., \qquad (6)$$
$$\Box_g DF \cong F \cdot DR + (R + DF)^2 + l.o..$$

  - $(R + DF)^2$ - quadratic terms.
  - $F \cdot D^2 F$, $F \cdot DR$ - first-order terms,
- Compare to vacuum case (K-R):

$$\Box_g R \cong R \cdot R.$$

The Kirchhoff-Sobolev Parametrix

- In $\mathbb{R}^{1+3}$, Kirchhoff's formula for scalar wave equations:

$$\Box\phi = \psi, \qquad \phi(p) \approx \int_{N^-(p)} \frac{1}{d(q,p)} \psi(q)\, d\sigma(q) + i.v.$$

- "Kirchhoff-Sobolev parametrix" (K-R): first-order generalization to curved spacetimes.
    - Valid on regular past null cones on a Lorentzian manifold (i.e., within null radius of injectivity).
    - Valid for covariant tensorial wave equations.
    - Supported entirely on past null cone.
    - Generalizable to covariant wave equations on arbitrary vector bundles (application: Y-M equations).

The Explicit Formula, Abridged

- Covariant wave equation $\Box_g \Phi = \Psi$.
- Transport equation on $\mathcal{N}^-(p)$:

$$D_L A = -\frac{1}{2} (\operatorname{tr} \chi) A, \qquad sA|_p = J_p,$$

  - A - tensor field on $\mathcal{N}^-(p)$, of same rank as Φ, Ψ - corresponds to r^{-1} in $\mathbb{R}^{1+3}$.
  - s - affine parameter (or another foliating function).
  - $\operatorname{tr} \chi$ - expansion of $\mathcal{N}^-(p)$.
- Kirchhoff-Sobolev parametrix given by

$$4\pi \cdot g \left(\Phi|_p , J_p \right) = \int_{\mathcal{N}^-(p)} [g(A, \Psi) + Error] + i.v..$$

Uniform Bounds in E-V

- Consider vacuum case, $\Box_g R \cong R^2$.
- To bound $\|R\|_{L^\infty}$, we must control principal term

$$\int_{\mathcal{N}^-(p)} |A| \, |R \cdot R| \, .$$

  - Main trick: the "Eardley-Moncrief" observation - one of the R's must be a flux component.
- Must also bound "error terms" and A.

Uniform Bounds in E-M

- Wave equations for R and DF.
- Quadratic terms $(R + DF)^2$ handled as in vacuum.
- However, cannot handle first-order terms this way.

Generalizing the Parametrix

- Alter the K-S parametrix to handle systems of covariant wave equations, with first-order terms.
- General form: for all $1 \leqslant m \leqslant n$,

$$\square_g ({}^m\Phi)_I + \sum_{c=1}^{n} ({}^{mc}P)_{\mu I}{}^J \cdot D^\mu ({}^c\Phi)_J = ({}^m\Psi)_I \, .$$

- Main idea: handle the ${}^{mc}P$'s through A, by altering the transport equation for A.
- Solve a *coupled system* of transport equations:

$$D_L ({}^mA)^I = -\frac{1}{2} (\mathrm{tr}\,\chi) ({}^mA)^I + \frac{1}{2} \sum_{c=1}^{n} ({}^{cm}P)_{LJ}{}^I ({}^cA)^J \, .$$

The Generalized Formula, Abridged

- Generalized formula given by:

$$4\pi \cdot \sum_{m=1}^{n} g\left(({}^m\Phi)|_p \, , ({}^m J_p) \right)$$

$$= \int_{\mathcal{N}^-(p)} \left[\sum_{m=1}^{n} g\left(({}^m A) \, , ({}^m \Psi)\right) + Error \right] + i.v..$$

- Used different proof than in K-R.
  - Avoids distributions.
  - Discretionary integration by parts - “never leaves the null cone.”
  - Avoids the optical function - weakens assumptions needed in K-R.
  - Gives initial value terms explicitly.
  - Again, can generalize to arbitrary vector bundles.

Uniform Bounds in E-M, Revisited

- In E-M case, $n = 2$, $({}^1\Phi) = R$, $({}^2\Phi) = DF$.
- $({}^{11}P)$ and $({}^{22}P)$ vanish, while $({}^{12}P)$, $({}^{21}P) \cong F$.
- L^∞-bounds for F, flux bounds for R and $DF \Rightarrow$ bounds for A.
- Remark: Generalization to vector bundles $\Rightarrow$ similar uniform bounds for Einstein-Yang-Mills.

More Issues

- To apply the K-S parametrix (E-V and E-M), we need:
  - Control for null injectivity radius.
  - Bounds for Ricci coefficients $\operatorname{tr}\chi$, $\hat{\chi}$, ζ, $\underline{\eta}$, $\operatorname{tr}\underline{\chi}$, $\underline{\hat{\chi}}$ on $\mathcal{N}^{-}(p)$, and their first derivatives.
- This is hard!
- Difficulty: we must control everything by L^2-quantities for R and DF, and by the breakdown criterion.
- Remark: We cannot similarly bound *causal* inj. radius. Thus, it is essential that the K-S parametrix depends only on null inj. radius.

A Basic Outline

- E-V: series of papers by K-R.
- Main task: extend to E-S and E-M settings.

1. Gigantic bootstrap: assume conditional bounds for Ricci coeff.
2. Assumptions for $\operatorname{tr}\chi \Rightarrow$ control null conj. radius
3. "Regularity" of time foliation $\Rightarrow$ control null inj. radius
4. Prove improved bounds for Ricci coeff.

- Remark: Must assume null injectivity radius to make full sense of $\operatorname{tr}\chi$, etc. on $\mathcal{N}^-(p)$.

Notes on Steps 2 and 3

- Step 2: Finiteness of $\operatorname{tr} \chi \Rightarrow$ null exponential map remains nonsingular.
- Step 3: Must control cut locus points.
  - Main tool: Existence of "almost Minkowski" coordinate systems $\Rightarrow \mathcal{N}^{-}(p)$ comparable to Minkowski cones.
  - At first cut locus point, show that distinct null geodesics intersect at angle π.

Discussion of Step 4

- Past results:
  - Klainerman, Rodnianski (2005): geodesic foliation, truncated null cones.
  - Q. Wang (2006): geodesic foliation, null cones.
  - D. Parlongue (2008): time foliation, truncated null cones.
  - Assume unit interval and small curvature flux, control Ricci coeff. by curvature flux (and time foliation).
- The nonvacuum analogue:
  - Time foliation, null cones.
  - Matter fields: control by both curvature and matter flux.
  - Assume small time interval and only bounded flux.

A Sample of the Results

- Main estimates:

$$\left\| \operatorname{tr}\chi - \frac{2}{t} \right\|_{L^\infty_\omega L^2_t} + \|\hat{\chi}\|_{L^\infty_\omega L^2_t} + \|\zeta\|_{L^\infty_\omega L^2_t} \lesssim 1,$$

$$\left\| \operatorname{tr}\chi - \frac{2}{t} \right\|_{\mathcal{H}^1} + \|\hat{\chi}\|_{\mathcal{H}^1} + \|\zeta\|_{\mathcal{H}^1} \lesssim 1,$$

- On sufficiently small segment $\mathcal{N}$ of $\mathcal{N}^-(p)$.
- Constant depends on flux and time foliation quantities.
- We also have the following:
  - $\| \operatorname{tr}\chi - 2t^{-1} \|_{L^\infty} \lesssim 1$.
  - Improved $\mathcal{H}^1$-estimates for $\operatorname{tr}\underline{\chi}$, $\underline{\hat{\chi}}$, $\underline{\eta}$ (uses recent results of Q. Wang: k satisfies tensor wave equation).

The Bootstrap Argument

- Assume main estimates hold on $\mathcal{N}$ of "small" length δ_0 with right-hand side replaced by "large" constant Δ_0.
- Conditional assumption: only when $\mathcal{N}$ remains regular, e.g., within the null injectivity radius.
- Show everything is $\lesssim \delta_0^{\frac{1}{2}} \Delta_0^2 + 1 \leqslant \Delta_0/2$.
- Main steps:
  - Integrate Raychaudhuri equation for $\operatorname{tr}\chi - 2t^{-1}$.
  - Integrate evolution equations for special derivative components $\nabla\!\!\!/ \operatorname{tr}\chi$, μ.
  - Elliptic estimates for $\nabla\!\!\!/ (\operatorname{tr}\chi)$, $\nabla\!\!\!/ \hat{\chi}$, $\nabla\!\!\!/ \zeta$.
  - Sharp trace estimates for $\hat{\chi}$, ζ.

Motivation for Sharp Trace Estimates

- In $I \times \mathbb{R}^2$, with I an interval, we have trace estimates

$$\left\| \partial_t f \right\|_{L^\infty_x L^2_t} \lesssim \left\| f \right\|_{H^2} ,$$

$$\left\| \int_I \partial_t f \cdot g|_{(t,\cdot)} \, dt \right\|_{B^0_{2,1}(\mathbb{R}^2)} \lesssim \left\| f \right\|_{H^1} \left\| g \right\|_{H^1} ,$$

  and other similar estimates.
- Goal: Prove similar tensorial estimates on $\mathcal{N}^-(p)$.
- Problems:
  - Cannot use classical Littlewood-Paley theory - not enough metric regularity.
  - Validity of estimates relies on bootstrap assumptions, i.e., *derivation of sharp trace estimates must be a part of the gigantic bootstrap argument for Ricci coefficients!*

The Main Estimates, Abridged

- K-R (2006): geometric L-P theory on manifolds.
  - Based on heat flow.
  - Can construct Besov spaces.
  - Can derive product estimates.
- Using L-P theory, derive sharp trace estimates:

$$\left\| \int_0^t \nabla\!\!\!/_L F \cdot G ds \right\|_{L^\infty_\omega B^0_{2,1}} \lesssim \|F\|_{\mathcal{H}^1} \left(\|G\|_{\mathcal{H}^1} + \|G\|_{L^\infty_\omega L^2_t} \right) .$$

  Other similar estimates hold.
- Major difficulty: Commutator estimates involving P_k.

Trace Estimates for $\hat{\chi}$, ζ

- Trace estimate: if $\nabla\!\!\!\!/ F = \nabla\!\!\!\!/_L P + E$, then

$$\|F\|_{L^\infty_\omega L^2_t} \lesssim \|F\|_{\mathcal{H}^1} + \|P\|_{\mathcal{H}^1} + \|E\|_{L^2_t B^0_{2,1}} .$$

  - Goal: Apply to $\hat{\chi}$, ζ.
- Problem: Not clear $\nabla\!\!\!\!/ \hat{\chi}$, $\nabla\!\!\!\!/ \zeta$ is of the above forms.
- Remark: Also need similar decompositions of $D \,\mathrm{Ric}$.
- To show this, we must use the following:
  - "Inverses" $\mathcal{D}^{-1}$ of elliptic Hodge operators.
  - Null Bianchi identities.
  - Commutators involving $\mathcal{D}^{-1}$.
  - Elaborate Besov estimates.

The End

Thank you!